Inverse conductivity problem on Riemann surfaces

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Abstract

An electrical potential U on a bordered Riemann surface X with conductivity function $\sigma > 0$ satisfies equation $d(\sigma d^c U) = 0$. The problem of effective reconstruction of σ from electrical currents measurements (Dirichlet-to-Neumann mapping) on the boundary: $U\big|_{bX} \mapsto \sigma d^c U\big|_{bX}$ is studied. We extend to the case of Riemann surfaces the reconstruction scheme given, firstly, by R.Novikov [N1] for simply connected X. We apply for this new kernels for $\bar\partial$ on the affine algebraic Riemann surfaces constructed in [H2].

0. Introduction

0.1. Inverse conductivity problem.

Let X be bordered oriented real two-dimensional manifold in $\mathbb{R}^3$ equiped with a smooth symmetric and positive tensor $\hat\sigma : T^*X \to T^*X$ on cotangent bundle T^*X, called anisotropic conductivity tensor on X, $\hat\sigma$ is called symmetric and positive if $\hat\sigma a \wedge b = \hat\sigma b \wedge a$ and $\hat\sigma a \wedge a > 0$ for any $a, b \in T^*X$.

Let u (correspondingly U) be a smooth function on bX (correspondingly on X) such that $U\big|_{bX} = u$, called electric potential on bX and correspondingly on X. 1-form $\hat\sigma dU$ on X is called electrical current on X. By Maxwell equation

$$d(\hat\sigma dU) = 0 \quad \text{on} \quad X.$$

Inverse conductivity problem consists in this case in the following: what kind of information about X and $\hat\sigma$ can be efficiently extracted from the knowledge of Dirichlet-to-Neumann mapping

$$u\big|_{bX} \mapsto \hat\sigma dU\big|_{bX} \ \forall\ u \in C^{(1)}(bX).$$

This important problem in the mathematical setting goes back to the inverse boundary values problems posed by I.M.Gelfand [G] and by A.P.Calderon [C].

This real problem is deeply related with complex analysis on Riemann surfaces. The first indication to this relation gives the following statement, obtaind at first by J.Sylvester [S] for simply connected X.

Under conditions above $\forall$ couple $(X, \hat\sigma)$ there exist a unique complex structure c on X and smooth scalar valued, positive conductivity function σ such that the equation $d(\hat\sigma dU) = 0$ takes the form $d(\sigma d^c U) = 0$, where $d^c = i(\bar\partial - \partial)$, $\sigma^2(x) = det\,\hat\sigma(x)$, $x \in X$.

This statement permits to reduce the inverse conductivity problem to the questions about reconstruction from Dirichlet-to-Neumann mapping of the genus of X, of the complex structure of X and of the scalar conductivity function σ on X.

These questions are well answered for the important case when X is a domain in $\mathbb{R}^2$, due to the sequence of works: [F1], [F2], [BC1], [SU], [N3], [N1], [GN], [BC2], [N2], [Na].

The exact reconstruction scheme for this case was discovered by R.Novikov [N1].

Formulated questions are well answered also for the case when conductivity function σ is known to be constant on X, i.e. when only Riemann surface X must be reconstructed from Dirichlet-to-Neumann data [LU], [Be], [HM].

In this paper we study another important case of this problem, when bordered two-dimensional manifold X and complex structure on X are known, but conductivity function σ on X must be reconstructed from Dirichlet-to-Neumann data.

0.2. Main results.

We extend here the R.Novikov's reconstruction scheme for the case of bordered Riemann surfaces. Our method (announced firstly in [H1]) is based on the appropriate new kernels for $\bar\partial$ on the affine algebraic Riemann surfaces constructed in [H2].

By this reason we use some special embedding of X into $\mathbb{C}^2$.

Let $\hat X$ be compactification of X such that $\hat X = \overline{X \cup X_0}$ be compact Riemann surface of genus g. Let $A = \{A_1, \ldots, A_d\}$ be divisor, generic, effective with support in X_0, consisting of $d = g + 2$ points.

By Riemann-Roch formula there exist three independent functions $f_0, f_1, f_2 \in \mathcal{M}(\hat X) \cap \mathcal{O}(\hat X \backslash A)$ having at most simple poles in the points of A. Without restriction of generality one can put $f_0 = const$. Let V be algebraic curve in $\mathbb{C}^2$ of the form

$$V = \{(z_1, z_2) \in \mathbb{C}^2 :\ z_1 = f_1(x),\ z_2 = f_2(x),\ x \in \hat X \backslash A\}.$$

Let $\tilde V$ be compactification of V in $\mathbb{C}P^2$ of the form $\tilde V = \{w \in \mathbb{C}P^2 :\ \tilde P(w) = 0\}$, where $\tilde P$ is homogeneous holomorphic polynomial of homogeneous coordinates $w = (w_0 : w_1 : w_2)$. Without loss of generality one can suppose: functions f_1, f_2 are such that

i) $\tilde V$ intersects $\mathbb{C}P^1_\infty = \{w \in \mathbb{C}P^2 :\ w_0 = 0\}$ transversally $\tilde V \cap \mathbb{C}P^1_\infty = \{a_1, \ldots, a_d\}$, where points $a_j = \big(0, 1, \lim\limits_{x \to A_j} \frac{f_2(x)}{f_1(x)}\big)$, $j = 1, 2, \ldots, d$ are different points of $\mathbb{C}P^1_\infty$.

ii) $V = \tilde V \backslash \mathbb{C}P^1_\infty$ is connected curve in $\mathbb{C}^2$ with equation $V = \{z \in \mathbb{C}^2 :\ P(z) = 0\}$, where $P(z) = \tilde P(1, z_1, z_2)$ such that $|\frac{\partial P}{\partial z_1}| \le const(V) |\frac{\partial P}{\partial z_2}|$, if $|z_1| \ge r_0 = const(V)$.

iii) For any $z^* \in V$, where $\frac{\partial P}{\partial z_2}(z^*) = 0$ we have $\frac{\partial^2 P}{\partial z_2^2}(z^*) \ne 0$.

With certain restriction of generality we suppose, in addition, that

iv) curve V is a regular curve, i.e. $grad\, P(z) \ne 0\ \forall\ z \in V$. This restriction must be eliminated in other publication.

Let us equip V by euclidean volume form $dd^c |z|^2\big|_V$.

Let $\varphi \mapsto f = \hat R \varphi$ be operator for solution of $\bar\partial f = \varphi$ on V from [H2], Proposition 2, $f \mapsto u = R_\lambda f$ be operator for solution of $(\partial + \lambda dz_1) u = f - \mathcal{H} f$ on V, where $\mathcal{H} f$ is projection of f on subspace of holomorphic (1,0)-forms on $\tilde V$ from [H2], Proposition 3, $\varphi \in L^\infty_{1,1} \cap L^1_{1,1}(V)$, $f \in W^{1,\tilde p}_{1,0}(V)$, $u \in W^{2,\tilde p}(V)$, $\tilde p > 2$.

Let $g_\lambda(z, \xi)$, $z, \xi \in V$, $\lambda \in \mathbb{C}$ be kernel of operator $R_\lambda \circ \hat R$ from [H2].

Let $V_X = \{(z_1, z_2) \in V :\ z_1 = f_1(x),\ z_2 = f_2(x),\ x \in X\}$.

Let σ be conductivity function on V with conditions $\sigma \in C^{(3)}(V)$, $\sigma > 0$ on V, $\sigma(z(x)) = \sigma(x)$, $x \in X$, $\sigma = const$ on $V \backslash V_X$.

Function $\psi(z, \lambda)$, $z \in V$, $\lambda \in \mathbb{C}$ will be called Faddeev type function on $V \times \mathbb{C}$ if $dd^c \psi = \frac{dd^c \sqrt{\sigma}}{\sqrt{\sigma}} \psi$ on V and $\forall \lambda \in \mathbb{C}$ $e^{-\lambda z} \psi(z, \lambda) \stackrel{\text{def}}{=} \mu(z, \lambda) \to 1$, $z \to \infty$, $|\bar\partial \mu| = O(\frac{1}{|z|+1})$, $z \in V$.

Theorem Under formulated conditions

I. There exists unique Faddeev type function $\psi(z, \lambda)$, $z \in V$, $\lambda \in \mathbb{C}$.

II. Function ψ and as a conseqence conductivity function σ can be reconstructed through Dirichlet-to-Neumann data by the following procedure.

II_a. From Dirichlet-to-Neumann data on bX by Proposition 3.1 (section 3) one can find restriction $\psi\big|_{bV_X}$ of the Faddeev type function $\psi(z,\lambda)$, $z \in V$, $\lambda \in \mathbb{C}$ as a unique solution of the Fredholm integral equation

$$\psi(z,\lambda)\big|_{bV_X} = e^{\lambda z_1} + \int\limits_{\xi \in bV_X} e^{\lambda(z_1-\xi_1)} g_\lambda(z,\xi) \cdot (\hat{\Phi}\psi(\xi) - \hat{\Phi}_0\psi(\xi)),$$

where $\hat{\Phi}\psi = \bar{\partial}\psi\big|_{bV_X}$, $\hat{\Phi}_0\psi = \bar{\partial}\psi_0\big|_{bV_X}$, $dd^c\psi_0\big|_{V_X} = 0$, $\psi_0\big|_{bV_X} = \psi\big|_{bV_X}$.

II_b. Using values of $\psi(z,\lambda)$ in arbitrary point $z^* \in bV_X$ by Proposition 2.2 (section 2) one can find "$\bar{\partial}$ scattering data":

$$b(\lambda) \stackrel{\text{def}}{=} \lim_{\substack{z\to\infty \\ z\in V}} \frac{\bar{z}_1}{\bar{\lambda}} e^{-\bar{\lambda}\bar{z}_1} \frac{\partial\psi}{\partial\bar{z}_1}(z,\lambda) = (\bar{\psi}(z^*,\lambda))^{-1} \frac{\partial\psi}{\partial\bar{\lambda}}(z^*,\lambda), \quad z^* \in bV_X$$

with estimate (2.12).

II_c. Using $b(\lambda)$, $\lambda \in \mathbb{C}$ by Proposition 2.3 (section 2) one can find values of

$$\mu(z,\lambda)\big|_{V_X} = \psi(z,\lambda)e^{-\lambda z_1}\big|_{V_X}, \quad \lambda \in \mathbb{C}$$

as a unique solution of Fredholm integral equation

$$\mu(z,\lambda) = 1 - \frac{1}{2\pi i} \int\limits_{\xi\in\mathbb{C}} b(\xi) e^{\bar{\xi}\bar{z}_1 - \xi z_1} \overline{\mu(z,\xi)} \frac{d\xi \wedge d\bar{\xi}}{\xi - \lambda}.$$

From equality $dd^c\psi = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\psi$ on X we find finally $\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$ on X.

Remarks

For the case $V = \mathbb{C}$ the reconstraction scheme I-II for potential q in the Schrödinger equation $-\Delta U + qU = EU$ on $X \subset V$ through the Dirichlet-to-Neumann data on bX was given for the first time by R.Novikov [N1]. However, in [N1] this scheme was rigorously justified only for the case when estimates of the type (2.12) are available, for example, if for $E \neq 0$ $\|q\| \leq const(E)$. By the additional result of A.Nachman [Na] the estimates of the type (2.12) are valid also if $E = 0$ and $q = \frac{\Delta\sqrt{\sigma}}{\sqrt{\sigma}}$, $\sigma > 0$, $\sigma \in C^{(2)}(X)$.

Part II_a in the present paper is completely similar to the related result of [N1] for $V = \mathbb{C}$.

Part II_b of this scheme for $V = \mathbb{C}$ is a consequence of works R.Beals, R.Coifman [BC1], P.Grinevich, S.Novikov [GN] and R.Novikov [N2].

Part II_c of this scheme follows from part II_b and the classical result of I.Vekua [V].

§1. Faddeev type function on affine algebraic Riemann surface. Uniqueness and existence

Let V be smooth algebraic curve in $\mathbb{C}^2$ defined in introduction, equiped by euclidean volume form $d\,d^c|z|^2\big|_V$.

Let $V_0 = \{z \in V : \ |z_1| \le r_0\}$, where r_0 satisfies condition ii) of introduction.

Definition

Let q be (1,1)-form in $C_{1,1}(\tilde{V})$ with support of q in V_0. For $\lambda \in \mathbb{C}$ function $z \mapsto \psi(z,\lambda)$, $z \in V$ will be called here the Faddeev type function associated with form (potential) q on V (and zero level of energy E) if

$$-d\,d^c\psi + q\psi = 0, \quad z \in V \tag{1.1}$$

and function $\mu = e^{-\lambda z_1}\psi$ satisfies the properties:

$$\mu \in C(\tilde{V}), \quad \lim_{\substack{z\to\infty \\ z\in V}} \mu(z,\lambda) = 1 \ \text{ and } \ |\bar{\partial}\mu| = O\big(\frac{1}{1+|z|}\big), \ z \in V.$$

From [F1], [F2], [N2] it follows that in the case $V = C = \{z \in \mathbb{C}^2 : \ z_2 = 0\}$, for almost all $\lambda \in \mathbb{C}$ the Faddeev type function $\psi = e^{\lambda z_1}\mu$ exists, unique and satisfies the Faddeev type integral equation

$$\mu(z,\lambda) = 1 + \frac{i}{2}\int\limits_{\xi\in V} g(z-\xi,\lambda)\mu(\xi,\lambda)q(\xi) \ \text{ where}$$

$$g(z,\lambda) = \frac{i}{2(2\pi)^2}\int\limits_{w\in\mathbb{C}} \frac{e^{i(w\bar{z}+\bar{w}z)}dw\wedge d\bar{w}}{w(\bar{w}-i\lambda)}$$

is so called the Faddeev-Green function for operator $\mu \mapsto \bar{\partial}(\partial + \lambda dz_1)\mu$ on $V = \mathbb{C}$.

Faddeev type functions $z \mapsto \psi(z,\lambda)$ are especially useful for solutions of inverse scattering or inverse boundary problems for equation (1.1) when such functions exist and unique for any $\lambda \in \mathbb{C}$. It was remarked by P.Grinevich and S.Novikov [GN] (see also [T]) that for some continuous q with compact support in $\mathbb{C}$ even with arbitrary small norm there exists the subset of exceptional λ^* for which the Faddeev type integral equation is not uniquely solvable.

From R.Novikov's works [N1], [N2] it follows that the Faddeev type functions associated with potential q on $V = \mathbb{C}$ and non-zero level of energy E exist $\forall\lambda \in \mathbb{C}$ if $\|q\| \le const(|E|)$.

From R.Beals, R.Coifman works [BC2] develloped by A.Nachman [Na] and R.Brown, G.Uhlmann [BU] it follows that for any potential q of the form $q = \frac{d\,d^c\sqrt{\sigma}}{\sqrt{\sigma}}$, where $\sigma \in C^2(\mathbb{C})$, $\sigma(z) \equiv const$ if $|z| \ge const$, Faddeev type function $z \mapsto \psi(z,\lambda)$ exists and unique for any $\lambda \in \mathbb{C}$.

Proposition 1.1 below gives the uniqueness of the Faddeev type function on affine algebraic Riemann surface V for potential $q = \frac{d\,d^c\sqrt{\sigma}}{\sqrt{\sigma}}$, where $\sigma \in C^2(V)$, $\sigma = const$ on $V\backslash V_X \subset V\backslash V_0$. The proof will be based on the approach going back to [BC2] in the case $V = \mathbb{C}$.

Proposition 1.1 (Uniqueness)

Let σ be positive function belonging to $C^{(2)}(V)$ such that $\sigma \equiv const > 0$ on

$$V\backslash V_X \subset V\backslash V_0 = \cup_{j=1}^{d} V_j,$$

where $\{V_j\}$ are connected components of $V\backslash V_0$.

Let $\mu \in L^{\infty}(V)$ such that $\frac{\partial \mu}{\partial z_1} \in L^{\tilde p}(V)$ for some $\tilde p > 2$ and μ satisfies equation

$$\bar\partial(\partial + \lambda dz_1)\mu = \frac{i}{2} q\mu \quad \text{where} \quad q = \frac{d\, d^c \sqrt{\sigma}}{\sqrt{\sigma}} \tag{1.2}$$

and for some $j \in \{1, 2, \ldots, d\}$ $\mu(z) \to 0$, $z \to \infty$, $z \in V_j$.

Then $\mu \equiv 0$.

Remark

Proposition 1 is still valid if to replace the condition $\frac{\partial \mu}{\partial z_1} \in L^{\tilde p}(V)$, $\tilde p > 2$ by the weaker condition $\partial \mu \in L^{\tilde p}(V)$.

Lemma 1.1

Put $f = e^{\lambda z_1}\mu$, $f_1 = \sqrt{\sigma}\frac{\partial f}{\partial z_1}$, $f_2 = \sqrt{\sigma}\frac{\partial f}{\partial \bar z_1}$, where μ satisfies conditions of Proposition 1.1. Then

$$d(\sigma d^c f) = 0 \quad \text{on} \quad V \quad \text{and} \tag{1.3}$$

$$\frac{\partial f_1}{\partial \bar z_1} = q_1 f_2, \quad \frac{\partial f_2}{\partial z_1} = \bar q_1 f_1, \tag{1.4}$$

where $q_1 = -\frac{\partial \log \sqrt{\sigma}}{\partial z_1}$. Besides, $q_1 \in L^p(V_0)$ $\forall\, p < 2$, $q_1 = 0$ on $V\backslash V_0$.

Proof

The property $q_1\big|_{V\backslash V_0} = 0$ follows from the property $\sigma \equiv const$ on $V\backslash V_0$. Put $V_0^{\pm} = \{z \in V_0 : \ \pm\big|\frac{\partial P}{\partial z_2}\big| \geq \pm\big|\frac{\partial P}{\partial z_1}\big|\}$. Put $\tilde q_1 = \frac{\partial \log \sqrt{\sigma}}{\partial z_2}$. Then $q_1\big|_{V_0^+} \in C^{(1)}(V_0^+)$ and $\tilde q_1\big|_{V_0^-} \in C^{(1)}(V_0^-)$. The identity $q_1\big|_{V_0^-} = \big(\frac{\partial z_1}{\partial z_2}\big)^{-1}\tilde q_1$ and property iii) imply that $q_1 \in L^p(V_0)$ $\forall p < 2$. Equation (1.3) for f is equivalent to the equation (1.2) for $\mu = e^{-\lambda z_1} f$. Equation (1.3) for f means that

$$\bar\partial F_1 + (\partial \ln \sqrt{\sigma}) \wedge F_2 = 0 \quad \text{and} \quad \partial F_2 + (\partial \ln \sqrt{\sigma}) \wedge F_1 = 0, \tag{1.5}$$

where $F_1 = \sqrt{\sigma} \cdot \partial f$ and $F_2 = \sqrt{\sigma} \cdot \bar\partial f$. Using z_1 as a local coordinate on V we obtain from (1.5) the system (1.4), where $f_1 = dz_1 \rfloor F_1$ and $f_2 = d\bar z_1 \rfloor F_2$.

Lemma 1.2

Put $m_1 = e^{-\lambda z_1} f_1$ and $m_2 = e^{-\lambda z_1} f_2$, where f_1, f_2 are defined in Lemma 4.1. Then system (1.3) is equivalent to system

$$\frac{\partial m_1}{\partial \bar z_1} = q_1 m_2, \quad \frac{\partial m_2}{\partial z_1} + \lambda m_2 = \bar q_1 m_1. \tag{1.6}$$

Besides,

$$m_1 = \sqrt{\sigma}\big(\lambda\mu + \frac{\partial\mu}{\partial z_1}\big) \text{ and } m_2 = \sqrt{\sigma}\frac{\partial\mu}{\partial\bar z_1}. \tag{1.7}$$

Proof

Putting in (1.4) $f_1 = e^{\lambda z_1}m_1$ and $f_2 = e^{\lambda z_1}m_2$, we obtain

$$\frac{\partial e^{\lambda z_1}m_1}{\partial\bar z_1} = q_1 e^{\lambda z_1}m_2 \Longleftrightarrow \frac{\partial m_1}{\partial\bar z_1} = q_1 m_2 \text{ and}$$
$$\frac{\partial e^{\lambda z_1}m_2}{\partial z_1} = \bar q_1 e^{\lambda z_1}m_1 \Longleftrightarrow \frac{\partial m_2}{\partial z_1} + \lambda z_2 = \bar q_1 m_1.$$

Besides,

$$f_1 = \sqrt{\sigma}\frac{\partial f}{\partial z_1} = \sqrt{\sigma}e^{\lambda z_1}\big(\lambda\mu + \frac{\partial\mu}{\partial z_1}\big) = e^{\lambda z_1}m_1,$$

where $m_1 = \sqrt{\sigma}\big(\lambda\mu + \frac{\partial\mu}{\partial z_1}\big)$ and

$$f_2 = \sqrt{\sigma}\frac{\partial f}{\partial\bar z_1} = \sqrt{\sigma}e^{\lambda z_1}\big(\frac{\partial\mu}{\partial\bar z_1}\big) = e^{\lambda z_1}m_2,$$

where $m_2 = \sqrt{\sigma}\frac{\partial\mu}{\partial\bar z_1}$.

Lemma 1.3

Put $u_\pm = m_1 \pm e^{-\lambda z_1+\bar\lambda\bar z_1}\bar m_2$. Then system (1.4) is equivalent to the system

$$\frac{\partial u_\pm}{\partial\bar z_1} = \pm q_1 e^{-\lambda z_1+\bar\lambda\bar z_1}\bar u_\pm.$$

Proof

From definition of $u_\pm$, using Lemma 1.2, we obtain

$$\frac{\partial u_\pm}{\partial\bar z_1} = \frac{\partial m_1}{\partial\bar z_1} \pm e^{-\lambda z_1+\bar\lambda\bar z_1}(\bar\lambda\bar m_2 - \bar\lambda\bar m_2 + q_1\bar m_1) =$$
$$q_1 m_2 \pm q_1 e^{-\lambda z_1+\bar\lambda\bar z_1}\bar m_1 = \pm q_1 e^{-\lambda z_1+\bar\lambda\bar z_1}(\bar m_1 \pm e^{\lambda z_1-\bar\lambda\bar z_1}m_2) =$$
$$\pm e^{-\lambda z_1+\bar\lambda\bar z_1}q_1\bar u_\pm.$$

Proof of Proposition 1.1

Let $u_\pm = m_1 \pm e^{-\lambda z_1+\bar\lambda\bar z_1}\bar m_2$. We will prove that under conditions of Proposition 1.1 $\forall\lambda \neq 0$ $u_\pm \equiv 0$ together with m_1 and m_2.

From equality $q_1 = -\frac{\partial\log\sqrt{\sigma}}{\partial z_1}$ and Lemma 1.3 we obtain that

$$\bar\partial u_\pm = \pm q_1 e^{-\lambda z_1+\bar\lambda\bar z_1}\bar u_\pm d\bar z_1 \in L^{\tilde p}_{0,1}(V) \cap L^1_{0,1}(V), \quad \tilde p > 2. \tag{1.8}$$

From (1.6), (1.7) we obtain that $m_1 \in L^{\tilde p}(V) \oplus L^\infty(V)$ and $\frac{\partial m_1}{\partial\bar z_1} \in L^p(V)$, $\forall p \geq 1$.

From (1.6), properties $\mu \in L^{\infty}(V)$, $\frac{\partial\mu}{\partial\bar{z}_1} \in L^{\tilde{p}}(V)$, $\mu(z) \to 0$, $z \in V_j$, $z \to \infty$ and from [H2] Corollary 1.1 we deduce that there exists

$$\lim_{\substack{z\to\infty\\ z\in V}} m_1(z) = \lim_{\substack{z\to\infty\\ z\in V}} \sqrt{\sigma}\big(\lambda\mu + \frac{\partial\mu}{\partial z_1}\big) = \lim_{\substack{z\to\infty\\ z\in V}} \sqrt{\sigma}\lambda\mu(z) = 0. \qquad (1.9)$$

From (1.8), (1.9) and generalized Liouville theorem from Rodin [R], Theorem 7.1, it follows that $u_{\pm} = 0$.

It means, in particular, that $\frac{\partial\mu}{\partial\bar{z}_1} = 0$, which together with (1.9) imply $\mu = 0$. Proposition 1.1 is proved.

Proposition 1.2 (Existence)

Let conductivity σ on V satisfies conditions of Proposition 1.1. Then $\forall\lambda \in \mathbb{C}$ there exists the Faddeev type function $\psi = \mu e^{\lambda z}$ on V associated with potential $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$, i.e.

$$a)\ \bar{\partial}(\partial + \lambda dz_1)\mu = \frac{i}{2}q\mu, \quad \text{where} \quad q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}},$$

$$b)\ \mu - 1 \in W^{1,\tilde{p}}(V) \cap C(\tilde{V}) \quad \forall\tilde{p} > 2,$$

$$\mu - 1 \in C^{(\infty)}(V\backslash V_0) \quad \text{and} \quad \mu - 1 = O\big(\frac{1}{|z_1|}\big), \quad z \in V\backslash V_0.$$

Moreover, $\forall\tilde{p} > 2$

$$c)\ \|\mu - 1\|_{L^{\tilde{p}}(V)} \le const(V,\sigma,\tilde{p})\{\min\big(\frac{1}{\sqrt{|\lambda|}}, \frac{1}{|\lambda|}\big)\},$$

$$\|\frac{\partial\mu}{\partial z_1}\|_{L^{\tilde{p}}(V)} \le const(V,\sigma,\tilde{p})\{\min(\sqrt{|\lambda|}, 1)\}, \quad \forall\tilde{p} > 2,$$

$$d)\ \forall\lambda\ \exists\ c \in \mathbb{C} \ \text{ such that } \ \big(\frac{\partial\mu}{\partial\bar{\lambda}} - c\big) \in W^{1,\tilde{p}}(V),$$

$$e)\ \text{under additional assumption} \ \ \sigma \in C^{(3)}(V)$$

$$\|\frac{\partial^2\mu}{\partial z_1^2}\|_{L^{\tilde{p}}(V)} \le const(V,\sigma) \cdot \lambda^{1/(1+1/\tilde{p})}.$$

Remark

Proposition 1.2 will be proved by approach going back to L.Faddeev [F1], [F2] and for the case $V = \mathbb{C}$ develloped in [N1], [N2], [Na].

Proof

Put $G_\lambda = R_\lambda \circ \hat{R}$, where $\hat{R}$ is operator, defined by formula (2.4) from Proposition 2 of [H2], and R_λ operator, defined by formula (3.1) from Proposition 3 of [H2]. Let function $\mu(z,\lambda)$ be such that $\forall\lambda$, $\mu \in W^{1,\tilde{p}}(V) \oplus const$ with respect to $z \in V$. Then $\forall\lambda \ne 0$ we have properties $q\mu \in C(V)$, $q\mu = 0$ on $V\backslash V_0$, which imply, in particular, that mapping $\mu \mapsto q\mu$ is compact operator from $W^{1,\tilde{p}}(V) \oplus const$ to $L^{\tilde{p}}(V) \cap C(V)$. By Lemmas 2.1, 2.2 and Proposition 2 from [H2] we have $\hat{R}q\mu \in W^{1,\tilde{p}}(V)$. Proposition 3 ii) implies that

$R_\lambda \circ \hat{R}(q\mu) \in W^{1,\tilde{p}}(V)$. Hence, mapping $\mu \mapsto \mu - R_\lambda \circ \hat{R}\frac{i}{2}q\mu$ is a Fredholm operator on $W^{1,\tilde{p}}(V) \oplus const$. By Proposition 1.1 the operator $I - R_\lambda \circ \hat{R}(\frac{i}{2}q\cdot)$ is invertible. Then there is unique function μ such that $(\mu - 1) \in W^{1,\tilde{p}}(V)$, $\forall \tilde{p} > 2$ and

$$\mu = 1 + R_\lambda \circ \hat{R}(\frac{i}{2}q\mu). \tag{1.10}$$

Let us check now statement a) of Proposition 1.2. From (1.10) and Proposition 3i) from [H2] we obtain

$$\begin{aligned}&(\partial + \lambda dz_1)\mu = \lambda dz_1 + (\partial + \lambda dz_1)R_\lambda \circ \hat{R}(\frac{i}{2}q\mu) = \\ &\lambda dz_1 + \mathcal{H}(\hat{R}(\frac{i}{2}q\mu)) + \hat{R}(\frac{i}{2}q\mu).\end{aligned} \tag{1.11}$$

From (1.11) and Proposition 2 of [H2] we obtain

$$\bar{\partial}(\partial + \lambda dz_1)\mu = \frac{i}{2}q\mu + \bar{\partial}(\lambda dz_1 + \mathcal{H}(\hat{R}(\frac{i}{2}q\mu)) = \frac{i}{2}q\mu,$$

where we have used that $\mathcal{H}(\hat{R}(\frac{i}{2}q\mu)) \in H_{1,0}(\tilde{V})$.

Property a) is proved.

For proving of properties b), c) it is sufficient to remark that by Proposition 3ii) and 3iii) from [H2] we have property c) and

$$\|(1 + |z_1|)(\mu - 1)\|_{L^\infty(V)} \le const(V, \sigma)\frac{1}{\sqrt{|\lambda|}}.$$

To prove property d) let us differentiate (1.10) with respect to $\bar{\lambda}$. We obtain

$$\begin{aligned}&\frac{\partial\mu}{\partial\bar{\lambda}} - R_\lambda \circ \big(\hat{R}(\frac{i}{2}q\frac{\partial\mu}{\partial\bar{\lambda}})\big) = \bar{z}_1\big(R_\lambda \circ \big(\hat{R}(\frac{i}{2}q\mu)\big)\big) - R_\lambda\big(\bar{\xi}_1\hat{R}(\frac{i}{2}q\mu)\big) = \\ &\bar{z}_1(\mu - 1) - R_\lambda\big(\bar{\xi}_1\hat{R}(\frac{i}{2}q\mu)\big).\end{aligned}$$

From Proposition 2 of [H2] we deduce that

$$\bar{\xi}_1\hat{R}(\frac{i}{2}q\mu) \in W^{1,\tilde{p}}_{1,0}(V) \oplus (const)dz_1.$$

From Proposition 3ii) of [H2] and Remark 1 to it we obtain

$$R_\lambda\big(\bar{\xi}_1\hat{R}(\frac{i}{2}q\mu)\big) \in W^{1,\tilde{p}}_{1,0}(V) \oplus const.$$

From Proposition 1.2 b) we deduce also

$$\bar{z}_1(\mu - 1) \in W^{1,\tilde{p}}(V) \oplus const.$$

Hence,

$$\frac{\partial\mu}{\partial\bar\lambda} = \big(I - R_\lambda \circ \hat R(\frac{i}{2}q\cdot)\big)^{-1}\big(\bar z_1(\mu-1) - R_\lambda(\bar\xi_1 \hat R(\frac{i}{2}q\mu)\big) \in W^{1,\tilde p}(V) \oplus (const).$$

Property e) follows (under condition $\sigma \in C^{(3)}(V)$) from Proposition 3iv) of [H2].

§2. **Equation** $\frac{\partial\mu(z,\lambda)}{\partial\bar\lambda} = b(\lambda)e^{\bar\lambda\bar z_1 - \lambda z_1}\bar\mu(z,\lambda),\ \lambda\in\mathbb{C}$

For further results it is important to obtain asymptotic development of $\frac{\partial\mu}{\partial\bar z_1}(z,\lambda)$ for $z_1 \to \infty$.

Proposition 2.1

Let conductivity function σ on V satisfy the conditions of Proposition 1.1. Let function $\psi(z,\lambda) = \mu(z,\lambda)e^{\lambda z}$ be the Faddeev type function on V associated with potential $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$. Then $\forall\lambda\in\mathbb{C}$ function $\frac{\partial\mu(z,\lambda)}{\partial\bar z_1}$ has the following asymptotic for $z_1\to\infty$:

$$e^{\lambda z_1 - \bar\lambda\bar z_1}\frac{\partial\mu}{\partial\bar z_1}\big|_{V_j} = \frac{B_{1,j}(\lambda)}{\bar z_1} + \sum_{k=2}^{\infty}\frac{B_{k,j}(\lambda)}{\bar z_1^k}, \tag{2.1}$$

where $|B_{1,j}(\lambda)| = O(\min(1,\sqrt{|\lambda|}))$, $j=1,2,\dots,d$ and

under additional condition $\sigma\in C^{(3)}(V)$ we have

$$e^{\lambda z_1 - \bar\lambda\bar z_1}\frac{\partial^2\mu}{\partial\bar z_1^2}\big|_{V_j} = \bar\lambda\frac{B_{1,j}}{\bar z_1} + \sum_{k=2}^{\infty}\frac{\bar\lambda B_{k,j}(\lambda) - (k-1)B_{k-1,j}(\lambda)}{\bar z_1^k}, \tag{2.2}$$

where $|B_{1,j}(\lambda)| = O(\min(\sqrt{|\lambda|}, |\lambda|^{-1/(1+\tilde p)}))$, $j = 1,2,\dots,d$ $\forall\tilde p>2$.

For proving of Proposition 2.1 we need the following decomposition statement for the Faddeev type function $\mu = \psi e^{-\lambda z}$ on $V\backslash V_0$.

Lemma 2.1

i) Let μ be function on $V\backslash V_0$, which satisfies equation

$$\bar\partial(\partial + \lambda dz_1)\mu = 0 \ \text{ on } \ V\backslash V_0 \tag{2.3}$$

and the property

$$(\mu-1)\big|_{V\backslash V_0} \in W^{1,\tilde p}(V\backslash V_0),\ \forall\tilde p>2.$$

Then

$$\begin{aligned} &A \overset{\text{def}}{=} \frac{\partial\mu}{\partial z_1} + \lambda\mu \in \mathcal{O}(\tilde V\backslash V_0);\quad A\big|_{V_j} = \lambda + \sum_{k=1}^{\infty} A_{k,j}\frac{1}{z_1^k},\\ &B \overset{\text{def}}{=} e^{-\lambda z_1 + \bar\lambda\bar z_1}\overline{\frac{\partial\mu}{\partial\bar z_1}} = \mathcal{O}(\tilde V\backslash V_0);\quad \bar B\big|_{V_j} = \sum_{k=1}^{\infty} B_{k,j}\frac{1}{\bar z_1^k}, \end{aligned} \tag{2.4}$$

$|z_1| > r_0$, $j = 1,2,\dots,d$.

ii) Let

$$M\big|_{V_j} = 1 + \sum_{k=1}^{\infty} \frac{a_{k,j}}{z_1^k} \quad \text{and} \quad \bar{N}\big|_{V_j} = 1 + \sum_{k=1}^{\infty} \frac{b_{k,j}}{\bar{z}_1^k}$$

be formal series with coefficients $a_{k,j}$ and $b_{k,j}$ determined by relations

$$\begin{aligned}
&\lambda a_{k,j} - (k-1)a_{k-1,j} = A_{k,j};\\
&\bar{\lambda} b_{k,j} - (k-1)b_{k-1,j} = B_{k,j};\ j = 1, 2 \ldots, d;\ k = 1, 2, \ldots
\end{aligned}$$

Then function μ has asymptotic decomposition

$$\begin{aligned}
&\mu\big|_{V_j} = M\big|_{V_j} + e^{\bar{\lambda}\bar{z}_1 - \lambda z_1}\bar{N}\big|_{V_j}, \quad z_1 \to \infty, \quad \text{i.e.} && (2.5)\\
&\mu\big|_{V_j} = M_\nu\big|_{V_j} + e^{\bar{\lambda}\bar{z}_1 - \lambda z_1}\bar{N}_\nu\big|_{V_j} + O\big(\frac{1}{|z_1|^{\nu+1}}\big), \quad \text{where}\\
&M_\nu\big|_{V_j} = 1 + \sum_{k=1}^{\nu} \frac{a_{k,j}}{z_1^k}, \quad \bar{N}_\nu\big|_{V_j} = \sum_{k=1}^{\nu} \frac{b_{k,j}}{\bar{z}_1^k}.
\end{aligned}$$

iii) Moreover, for any $A \in \mathcal{O}(\tilde{V}\backslash V_0)$, $A(z) \to \lambda$, $z \to \infty$, there exist $B \in \mathcal{O}(\tilde{V}\backslash V_0)$, $B(z) \to 0$, $z \to \infty$, and μ such that μ satisfies (2.3), (2.4) and $(\mu - 1) \in W^{1,\tilde{p}}(V\backslash V_0)$.

Proof

i) From equation (2.3) it follows that

$$\partial\bar{\partial}(e^{\lambda z_1}\mu(z,\lambda))\big|_{V\backslash V_0} = 0.$$

It means that $\bar{\partial}(e^{\lambda z_1}\mu(z,\lambda)) = e^{\lambda z_1}\bar{\partial}\mu$ is antiholomorphic form on $V\backslash V_0$ and $(\partial\mu + \lambda\mu dz_1)$ is holomorphic form on $V\backslash V_0$. From this, condition $\bar{\partial}\mu \in L^{\tilde{p}}_{0,1}(V\backslash V_0)$ and the Cauchy theorem it follows that

$$\begin{aligned}
&e^{\lambda z_1}\bar{\partial}\mu\big|_{V_j} = e^{\bar{\lambda}\bar{z}_1}\bar{B}d\bar{z}_1\big|_{V_j} = e^{\bar{\lambda}\bar{z}_1}\sum_{k=1}^{\infty}\frac{B_{k,j}}{\bar{z}_1^k}d\bar{z}_1\big|_{V_j}\\
&\text{and} \quad (\partial\mu + \lambda\mu dz_1)\big|_{V_j} = A dz_1\big|_{V_j} = \big(\lambda + \sum_{k=1}^{\infty}\frac{A_{k,j}}{z_1^k}\big)dz_1\big|_{V_j}.
\end{aligned}$$

ii) From (2.4), (2.5) we obtain, at first, that

$$\bar{\partial}\mu = e^{-\lambda z_1}\bar{\partial}(e^{\bar{\lambda}\bar{z}_1}\bar{N}_\nu) + O\big(\frac{1}{|\bar{z}_1|^{\nu+1}}\big)$$

and then

$$\mu = M_\nu + e^{\bar{\lambda}\bar{z}_1 - \lambda z_1}\bar{N}_\nu + O\big(\frac{1}{|z_1|^{\nu+1}}\big). \tag{2.6}$$

iii) Let

$$A\big|_{V_j} = \lambda + \sum_{k=1}^{\infty} \frac{A_{k,j}}{z_1^k} \in L^{\tilde p}(V\backslash V_0).$$

Proposition 3 iii) from [H2], applied to (1,0) forms on the complex plan with coordinate z_1, gives that $\forall \lambda \neq 0$ there exists $\mu \in W^{1,\tilde p}(V\backslash V_0) \oplus 1$ such that $A = \frac{\partial \mu}{\partial z_1} + \lambda\mu$. Put $\bar B = e^{\lambda z_1 - \bar\lambda \bar z_1} \frac{\partial \mu}{\partial \bar z_1}$. Then $\bar B \in L^{\tilde p}(V\backslash V_0)$ and

$$\frac{\partial \bar B}{\partial z_1} = e^{\lambda z_1 - \bar\lambda \bar z_1} \big(\frac{\partial}{\partial \bar z_1}(\lambda\mu + \frac{\partial \mu}{\partial z_1})\big) = e^{\lambda z_1 - \bar\lambda \bar z_1} \big(\frac{\partial A}{\partial \bar z_1}\big) = 0,$$

i.e. $B \in \mathcal{O}(\tilde V\backslash V_0)$. By construction μ satisfies (2.3), (2.4) and $(\mu - 1) \in W^{1,\tilde p}$.

Lemma 2.2
Functions M_ν and N_ν from decomposition (2.6) have the following properties:

$$\forall\, z \in \tilde V\backslash V_0 \;\; \exists \lim_{\nu\to\infty} \big(\frac{\partial M_\nu}{\partial z_1} + \lambda M_\nu\big) \stackrel{\text{def}}{=} \frac{\partial M}{\partial z_1} + \lambda M \;\;\text{and}$$
$$\exists \lim_{\nu\to\infty} \big(\frac{\partial N_\nu}{\partial z_1} + \lambda N_\nu\big) \stackrel{\text{def}}{=} \frac{\partial N}{\partial z_1} + \lambda N.$$

Functions $\frac{\partial M}{\partial z_1} + \lambda M$ and $\frac{\partial N}{\partial z_1} + \lambda N$ belongs to to $\mathcal{O}(\tilde V\backslash V_0)$ and

$$\frac{\partial \mu}{\partial \bar z_1} = e^{\bar\lambda \bar z_1 - \lambda z_1} \big(\frac{\partial \bar N}{\partial \bar z_1} + \bar\lambda \bar N\big), \tag{2.7}$$
$$\frac{\partial \mu}{\partial z_1} + \lambda\mu = \frac{\partial M}{\partial z_1} + \lambda M, \tag{2.8}$$
$$\frac{\partial N}{\partial z_1} + \lambda N \to 0, \;\; \text{if} \;\; z_1 \to \infty.$$

Proof
Let us show that (2.4) implies (2.7) and (2.8). Indeed,

$$\frac{\partial \mu}{\partial \bar z_1} = \lim_{\nu\to\infty} \big(\bar\lambda e^{\bar\lambda \bar z_1 - \lambda z_1} \bar N_\nu + e^{\bar\lambda \bar z_1 - \lambda z_1} \frac{\partial \bar N_\nu}{\partial \bar z_1}\big) = e^{\bar\lambda \bar z_1 - \lambda z_1} \lim_{\nu\to\infty} \big(\frac{\partial \bar N_\nu}{\partial \bar z_1} + \bar\lambda \bar N_\nu\big) =$$
$$e^{\bar\lambda \bar z_1 - \lambda z_1} \big(\frac{\partial \bar N}{\partial \bar z_1} + \bar\lambda \bar N\big),$$
$$\frac{\partial \mu}{\partial z_1} + \lambda\mu = \lim_{\nu\to\infty} \left(\frac{\partial M_\nu}{\partial z_1} - \lambda e^{\bar\lambda \bar z_1 - \lambda z_1} \bar N_\nu + \lambda M_\nu + \lambda e^{\bar\lambda \bar z_1 - \lambda z_1} \bar N_\nu\right) =$$
$$\lim_{\nu\to\infty} \big(\frac{\partial \bar M_\nu}{\partial z_1} + \lambda M_\nu\big) = \frac{\partial M}{\partial z_1} + \lambda M.$$

Properties (2.6), (2.7), (2.8), $\bar\partial \mu \in L^{\tilde p}_{0,1}(V)$ and Riemann extension theorem imply that $\frac{\partial M}{\partial z_1} + \lambda M$ and $\frac{\partial N}{\partial z_1} + \lambda N$ belongs to $\mathcal{O}(\tilde V\backslash V_0)$ and $\frac{\partial N}{\partial z_1} + \lambda N \to 0$, if $z_1 \to \infty$.

Corollary

In conditions of Lemmas 2.1, 2.2 we have convergence $M_\nu \to M$ and $N_\nu \to N$, $\nu \to \infty$, in general, only in the space of formal series, in spite of that convergence $\frac{\partial M_\nu}{\partial z_1} + \lambda M_\nu \to \frac{\partial M}{\partial z_1} + \lambda M$ and $\frac{\partial N_\nu}{\partial z_1} + \lambda N_\nu \to \frac{\partial N}{\partial z_1} + \lambda N$ take place in the space $\mathcal{O}(\tilde{V}\backslash V_0)$.

Simple example

Put $V_0 = \{z \in V: \ |z_1| < 1\}$, $\lambda = 1$, $A = \lambda + \sum\limits_{k=1}^{\infty} \frac{A_k}{z_1^k}$ with $A_k = 1$, $k = 1, 2, \ldots$

By Lemma 2.1 there exists $\mu = M + e^{\bar{\lambda}\bar{z}_1 - \lambda z_1}\bar{N}$, which satisfies (2.3), (2.4), where $M = 1 + \sum\limits_{k=1}^{\infty} \frac{a_k}{z_1^k}$ with a_k determined by relations $a_k - (k-1)a_{k-1} = A_k = 1$; $k = 1, 2, \ldots$. It gives $a_k = (k-1)!\big(1 + \frac{1}{2!} + \ldots + \frac{1}{(k-1)!}\big)$. We have $|a_k|^{1/k} \to \infty$, $k \to \infty$, i.e. radius of convergence of serie for M is equal to zero, in spite of that $|A_k|^{1/k} = 1$, $k = 1, 2, \ldots$.

Proof of Proposition 2.1

Estimate for $\frac{\partial \mu}{\partial z_1}$ from Proposition 1.2 c) and the Cauchy theorem applied to antiholomorphic function $e^{\lambda z_1 - \bar{\lambda}\bar{z}_1}\frac{\partial \mu}{\partial \bar{z}_1}$ implies development (2.1) and estimate

$$|B_{1,j}(\lambda)| = O(\min(1, \sqrt{|\lambda|})). \tag{2.9}$$

Estimate for $\frac{\partial^2 \mu}{\partial \bar{z}_1^2}$ from Proposition 1.2 e) and the Cauchy theorem for antiholomorphic in z_1 function $e^{\lambda z_1 - \bar{\lambda}\bar{z}_1}\frac{\partial^2 \mu}{\partial \bar{z}_1^2}\big|_{V_j}$ imply development (2.2) and estimate

$$|B_{1,j}(\lambda)| = O(\min(\sqrt{|\lambda|}, |\lambda|^{-1/(1+\tilde{p})})). \tag{2.10}$$

Proposition 2.1 is proved.

The next proposition gives $\bar{\partial}$-equation on Faddeev function $\mu(z,\lambda)$ with respect to parameter $\lambda \in \mathbb{C}$. For the case $V = \mathbb{C}$ this proposition goes back to Beals, Coifmann [BC1], Grinevich, S.Novikov [GN] and R.Novikov [N2].

Proposition 2.2

Let conductivity function σ on V satisfy the conditions of Proposition 1.1. Let function $\psi(z,\lambda) = \mu(z,\lambda)e^{\lambda z}$ be the Faddeev type function on V constructed in Proposition 1.2 and associated with potential $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$. Then $\forall z \in V$ the following $\bar{\partial}$-equation with respect to $\lambda \in \mathbb{C}$ takes place

$$\frac{\partial \mu}{\partial \bar{\lambda}} = b(\lambda)e^{\bar{\lambda}\bar{z}_1 - \lambda z_1}\bar{\mu}, \tag{2.11}$$

where

$$b(\lambda) \stackrel{\text{def}}{=} \lim_{\substack{z\to\infty\\ z\in V}} \frac{\bar{z}_1}{\bar{\lambda}} e^{\lambda z_1 - \bar{\lambda}\bar{z}_1}\frac{\partial \mu}{\partial \bar{z}_1},$$

$$|b(\lambda)| \le const(V,\sigma)\{\min\big(\frac{1}{\sqrt{|\lambda|}}, \frac{1}{|\lambda|}\big)\} \ \text{ if } \ \sigma \in C^{(2)}(V) \ \text{ and} \tag{2.12}$$

$$|b(\lambda)| \le const(V,\sigma,\tilde{p})\{\min\big(\frac{1}{\sqrt{|\lambda|}}, \big(\frac{1}{|\lambda|}\big)^{1+1/\tilde{p}}\big)\} \ \text{ if } \ \sigma \in C^{(3)}(V), \ \forall \tilde{p} > 2.$$

Remark

The proof below is a generalization of the R.Novikov proof [N2] of the corresponding statement for the case $V = \mathbb{C}$.

Proof

Equation $dd^c\psi = q\psi$ for the Faddeev type function $\psi = \mu \cdot e^{\lambda z}$ is equivalent to the equation $\bar\partial(\partial + \lambda dz_1)\mu = \frac{i}{2}q\mu$. Put $\psi_\lambda = \partial\psi/\partial\bar\lambda$ and $\mu_\lambda = \partial\mu/\partial\bar\lambda$. By Proposition 1.2 d) $\forall\lambda \in \mathbb{C}$ function μ_λ belongs to $W^{1,\tilde p}(V) \oplus const(\lambda)$. Besides, from equation $dd^c\psi = q\psi$ it follows the equation

$$dd^c\psi_\lambda = q\psi_\lambda \quad \text{on} \quad V.$$

From Lemmas 2.1, 2.2, Proposition 2.1 and Proposition 1.2 b),c) we obtain

$$\begin{aligned} &\frac{\partial\mu}{\partial\bar z_1}\Big|_{V_j} = e^{\bar\lambda\bar z_1 - \lambda z_1}\frac{B_{1,j}(\lambda)}{\bar z_1} + O\big(\frac{1}{|z_1|^2}\big) \quad \text{and} \\ &\left(\frac{\partial\mu}{\partial z_1} + \lambda\mu\right)\Big|_{V_j} = \lambda + \frac{A_{1,j}(\lambda)}{z_1} + O\big(\frac{1}{|z_1|^2}\big). \end{aligned} \tag{2.13}$$

From (2.4)-(2.6) and (2.13) we deduce that

$$\mu = 1 + \frac{a_j(\lambda)}{z_1} + e^{\bar\lambda\bar z_1 - \lambda z_1}\frac{b_j(\lambda)}{\bar z_1} + O\big(\frac{1}{|z_1|^2}\big), \tag{2.14}$$

$$\text{where} \quad \bar\lambda b_j(\lambda) = B_{1,j}, \quad \lambda a_j(\lambda) = A_{1,j}, \quad j = 1, 2, \ldots, d. \tag{2.15}$$

From (2.14) with help of Proposition 1.2 d) we obtain

$$\begin{aligned} &\psi = e^{\lambda z_1}\mu = e^{\lambda z_1}\big(1 + \frac{a_j(\lambda)}{z_1} + e^{\bar\lambda\bar z_1 - \lambda z_1}\frac{b_j(\lambda)}{\bar z_1} + O\big(\frac{1}{|z_1|^2}\big)\big), \\ &\text{and} \quad \psi_\lambda = \frac{\partial\psi}{\partial\bar\lambda} = e^{\bar\lambda\bar z_1}\big(b_j(\lambda) + O\big(\frac{1}{|z_1|}\big)\big), \quad z \in V_j. \end{aligned}$$

Put $\mu_\lambda = e^{-\lambda z_1}\psi_\lambda$. We obtain $\bar\partial(\partial + \lambda dz_1)\mu_\lambda = q\mu_\lambda$ and

$$\mu_\lambda = e^{\bar\lambda\bar z_1 - \lambda z_1}\big(b_j + O\big(\frac{1}{|z_1|}\big)\big), \quad z \in V_j.$$

Proposition 1.1 about uniqueness of the Faddeev type function implies equality (2.11), where

$$b(\lambda) \stackrel{\text{def}}{=} b_1(\lambda) = \ldots = b_d(\lambda). \tag{2.16}$$

We have also equalities

$$B_{1,1}(\lambda) = \ldots = B_{1,d}(\lambda).$$

Put

$$B(\lambda) \stackrel{\text{def}}{=} B_{1,1}(\lambda) = \ldots = B_{1,d}(\lambda).$$

Estimates (2.12) follow from equalities (2.15), (2.16) and inequalities (2.9), (2.10).

Proposition 2.3

Let, under conditions of Proposition 2.2, conductivity function $\sigma \in C^{(3)}(V)$. Then $\forall\, z \in V$ function $\lambda \mapsto \mu(z,\lambda)$, $\lambda \in \mathbb{C}$,is a unique solution of the following integral equation

$$\mu(z,\lambda) = 1 - \frac{1}{2\pi i} \int\limits_{\xi\in\mathbb{C}} b(\xi) e^{\bar\xi\bar z_1 - \xi z_1} \overline{\mu(z,\xi)} \frac{d\xi \wedge d\bar\xi}{\xi - \lambda}, \tag{2.17}$$

where function $\lambda \mapsto b(\lambda)$ from (2.11) belongs to $L^q(\mathbb{C})$ $\forall\, q \in (4/3, 4)$.

Proof

Estimates (2.12) imply that $\forall\, q \in (4/3,4)$ function $b(\lambda)$ from Proposition 2.2 belongs to $L^q(\mathbb{C})$. By the classical Vekua result [Ve] with such functions $b(\lambda)$ the equation (2.17) is the uniquely solvable Fredholm integral equation in the space $C(\bar{\mathbb{C}})$.

§3. **Reconstruction of Faddeev type function ψ and of conductivity σ on X through Dirichlet to Neumann data on bX**

Let X be domain with smooth (de class $C^{(2)}$) boundary on V such that $X \supseteq \bar V_0$. Let $\sigma \in C^{(3)}(V)$, $\sigma > 0$ on V and $\sigma \equiv const$ on $V\backslash X$. Let $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$, then $q \in C^{(1)}_{1,1}(X)$ and $supp\, q \subseteq X$.

Definition (Dirichlet-Neumann operator)

Let $u \in C^{(1)}(bX)$ and $U \in W^{1,\tilde p}(X)$, $\tilde p > 2$ be solution of the Dirichlet problem $d\sigma d^c U\big|_X = 0$, $U\big|_{bX} = u$, where $d^c = i(\bar\partial - \partial)$. Operator $u\big|_{bX} \to \sigma d^c U\big|_{bX}$ is called usually by Dirichlet to Neumann operator. Put $\tilde\psi = \sqrt{\sigma}U$ and $\psi = \sqrt{\sigma}u$. Then

$$dd^c\tilde\psi = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\tilde\psi = q\tilde\psi \quad \text{on} \quad X. \tag{3.1}$$

Operator $\psi\big|_{bX} \mapsto \bar\partial\tilde\psi\big|_{bX}$ will be called here by Dirichlet-Neumann operator. The data contained in operator $u\big|_{bX} \to \sigma d^c U\big|_{bX}$ and in $\psi\big|_{bX} \to \bar\partial\tilde\psi\big|_{bX}$ are equivalent, but for our further statements operator $\psi\big|_{bX} \to \bar\partial\tilde\psi\big|_{bX}$ is more convenient.

Let ψ_0 be solution of Dirichlet problem

$$dd^c\psi_0 = 0, \quad \psi_0\big|_{bX} = \psi\big|_{bX}.$$

Put

$$\hat\Phi\psi = \bar\partial\tilde\psi\big|_{bX} \quad \text{and} \quad \hat\Phi_0\psi = \bar\partial\tilde\psi_0\big|_{bX}, \tag{3.2}$$

where operator $\psi\big|_{bX} \to \hat\Phi_0\psi$ is Dirichlet-Neumann operator for equation (3.1) with potential $q \equiv 0$.

Proposition 3.1 (Reconstruction of $\psi\big|_{bX}$ through Dirichlet-Neumann data)

Let $\psi = \mu(z,\lambda)\cdot e^{\lambda z_1}$ be the Faddeev function associated with potential $q = \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}$. Then $\forall \lambda \in \mathbb{C}\backslash\{0\}$ the restriction $\psi\big|_{bX}$ of ψ on bX is a unique solution in $C(bX)$ of the Fredholm integral equation:

$$\psi(z,\lambda)\big|_{bX} = e^{\lambda z_1} - \int\limits_{\xi\in bX} e^{\lambda(z_1-\xi_1)} g_\lambda(z,\xi)\cdot(\hat\Phi\psi(\xi) - \hat\Phi_0\psi(\xi)), \tag{3.3}$$

where $g_\lambda(z,\xi)$ - kernel of operator $R_\lambda \circ \hat{R}$,

$$\hat{\Phi}\psi(\xi) - \hat{\Phi}_0\psi(\xi) = \int\limits_{w\in bX} (\Phi(\xi,w) - \Phi_0(\xi,w))\psi(w),$$

$\Phi(\xi,w)$, $\Phi_0(\xi,w)$ are kernels of operators $\hat{\Phi}$ and $\hat{\Phi}_0$.

Remark

This proposition for the case $V = \mathbb{C}$ coincide with the second part of Theorem 1 from [N1].

Lemma 3.1 (Green-Riemann formula)

$\forall\ f, g \in C^{(1)}(X)$ we have equality

$$\int\limits_X g \wedge \partial\bar{\partial} f - \int\limits_X f \wedge \partial\bar{\partial} g = \int\limits_{bX} g \wedge \bar{\partial} f + \int\limits_{bX} f \wedge \partial g.$$

Proof

$$\int\limits_X g \wedge \partial\bar{\partial} f - \int\limits_X f \wedge \partial\bar{\partial} g = \int\limits_X g \wedge \partial\bar{\partial} f + \int\limits_X f \wedge \bar{\partial}\partial g =$$
$$\int\limits_{bX} g \wedge \bar{\partial} f - \int\limits_X \partial g \wedge \bar{\partial} f + \int\limits_{bX} f \wedge \partial g - \int\limits_X \bar{\partial} f \wedge \partial g =$$
$$\int\limits_{bX} g \wedge \bar{\partial} f + \int\limits_{bX} f \wedge \partial g.$$

Lemma 3.2

Let $\psi = e^{\lambda z_1}\mu$ be the Faddeev function associated with $q \in C_{1,1}(V)$, $supp\, q \subseteq X$. Let $G_\lambda(z,\xi) = e^{\lambda(z_1-\xi_1)} g_\lambda(z,\lambda)$, where $g_\lambda(z,\xi)$ - kernel of operator $R_\lambda \circ \hat{R}$. Then $\forall\ z \in V\backslash X$ we have equality

$$\psi(z,\lambda) = e^{\lambda z_1} - \int\limits_{\xi\in bX} G_\lambda(z,\xi)\bar{\partial}\psi(\xi) - \int\limits_{\xi\in bX} \psi(\xi)\partial G_\lambda(z,\xi). \tag{3.4}$$

Proof

From definition of the Faddeev function $\psi = e^{\lambda z_1}\mu$ we have

$$\psi(z,\lambda) = e^{\lambda z_1} + \hat{G}_\lambda\big(\frac{i}{2}q\psi\big) \ \text{ on } \ V, \tag{3.5}$$

where $\hat{G}_\lambda$ - operator with kernel $G_\lambda(z,\xi)$ and $dd^c\psi = q\psi$. Besides,

$$\int\limits_X G_\lambda\big(\frac{i}{2}q\psi\big) = -\int\limits_X G_\lambda \partial\bar{\partial}\psi.$$

Using the Green-Riemann formula from Lemma 3.1 we have

$$\int\limits_X G_\lambda \partial\bar\partial\psi = \int\limits_X \psi\partial\bar\partial G_\lambda + \int\limits_{bX} G_\lambda\bar\partial\psi + \int\limits_{bX} \psi\partial G_\lambda.$$

For $z \in V\backslash X$ we have $\partial\bar\partial G_\lambda = 0$. Hence,

$$-\int\limits_X G_\lambda(\frac{i}{2}q\psi) = \int\limits_{bX} G_\lambda\bar\partial\psi + \int\limits_{bX} \psi\partial G_\lambda. \tag{3.6}$$

From (3.5) and (3.6) we obtain (3.4).

Proof of Proposition 3.1

Let $\psi_0 : \ \bar\partial\partial\psi = 0$ and $\psi_0\big|_{bX} = \psi$. Then by Lemma 3.1 $\forall\ z \in V\backslash X$ we have

$$\int\limits_{bX} \psi_0\partial G_\lambda + \int\limits_{bX} G_\lambda\bar\partial\psi_0 = 0. \tag{3.7}$$

Combining this relation with (3.4) we obtain

$$\psi(z,\lambda) = e^{\lambda z_1} - \int\limits_{bX} G_\lambda(z,\xi)(\bar\partial\psi(\xi) - \bar\partial\psi_0(\xi)). \tag{3.8}$$

Equality (3.8) implies (3.3), where $\hat\Phi\psi$, $\hat\Phi_0\psi$ are defined by (3.2). Integral equation (3.3) is the Fredholm equation in $L^\infty(bX)$, because operator $(\hat\Phi - \hat\Phi_0)$ is a compact operator in $L^\infty(bX)$. For the case $V = \mathbb{C}$ Proposition 1 of [N1], contains explicit inequality:

$$\lim_{\substack{w\to\xi \\ w,\xi\in bX}} \frac{|\Phi(\xi,w) - \Phi_0(\xi,w)|}{|\ln|\xi - w||} \le (const)\Big|\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\Big|(w). \tag{3.9}$$

Because of its local nature, equality (3.9) is valid for our more general case.

Existence of the Faddeev function $\forall\lambda \ne 0$, proved in Proposition 1.2, implies existence of solution of equation (3.3). The uniqueness of solution of equation (3.3) we deduce (as in proof of Proposition 2 in [N1]) from the following statement.

Lemma 3.3

Let $q \in C_{1,1}(V)$, $supp\, q \subseteq X$. Then each solution $\psi = \psi(z,\lambda)$ of equation $dd^c\psi = q\psi$ on V which coincides on bX with solution of integral equation (3.3) in space $C(bX)$ is the Faddeev type function associated with q.

Proof

Let ψ satisfy $dd^c\psi = q\psi$ on V, $\psi\big|_{bX} \in C(bX)$ and $\psi\big|_{bX}$ satisfy integral equation (3.3)=(3.8). From (3.8) with help of (3.7) we obtain formula (3.4) for $\psi(z,\lambda)$, $z \in V\backslash X$,

$\lambda \in \mathbb{C}$. From the Sohotsky-Plemelj jump formula we deduce (see [HM], Lemma 15) that $\forall\ z^* \in bX$

$$\psi(z^*) = \lim_{\substack{z\to z^*\\ z\in X}} \Big(\int\limits_{bX} G_\lambda\bar\partial\psi + \psi\partial G_\lambda\Big) - \lim_{\substack{z\to z^*\\ z\in V\backslash X}} \Big(\int\limits_{bX} G_\lambda\bar\partial\psi + \psi\partial G_\lambda\Big). \tag{3.10}$$

From (3.4) and (3.10) we obtain equality

$$e^{\lambda z_1} = \int\limits_{bX} G_\lambda\bar\partial\psi + \psi\partial G_\lambda \quad \text{for} \quad z \in X, \quad \lambda \in \mathbb{C}. \tag{3.11}$$

Using the Green-Riemann formula (Lemma 3.1) we obtain further

$$\begin{gathered}\int\limits_{bX} G_\lambda\bar\partial\psi + \psi\partial G_\lambda = \int\limits_{X} \psi\bar\partial\partial G_\lambda - \int\limits_{X} G_\lambda\bar\partial\partial\psi = \\ \begin{cases} \psi(z) - \int\limits_X G_\lambda\bar\partial\partial\psi, & \text{if } z \in X,\ \lambda \in \mathbb{C} \\ \quad -\int\limits_X G_\lambda\bar\partial\partial\psi, & \text{if } z \in V\backslash X,\ \lambda \in \mathbb{C} \end{cases}\end{gathered} \tag{3.12}$$

Equalities (3.4), (3.11) and (3.12) imply

$$\psi(z) = e^{\lambda z_1} + \int\limits_{V} G_\lambda\bar\partial\partial\psi = e^{\lambda z_1} + \hat G_\lambda(\frac{i}{2}q\psi),$$

i.e. ψ satisfies (3.5). From equality (3.5) and Proposition 3 ii) from [H2] we obtain that $\mu - 1 = \psi e^{-\lambda z_1} - 1 \in W^{1,\tilde p}(V)$ and ψ is the Faddeev type function associated with q.

Lemma 3.3 is proved.

The uniqueness of solution of equation (3.3) in $C(bX)$ follows now from Lemma 3.3 and the uniqueness of the Faddeev type function associated with q proved in Proposition 1.1.

Proposition 3.1 is proved.

Proposition 3.2 (Reconstruction of $\psi\big|_X$ through $\psi\big|_{bX}$)

In conditions of Propositions 1.1, 1.2 the following properties for "$\bar\partial$-scattering data $b(\lambda)$" permit to reconstruct $\psi\big|_X$ through $\psi\big|_{bX}$

$$\forall\ z^* \in bX\ \exists \lim_{\substack{z\to\infty\\ z\in V}} \frac{\bar z_1}{\bar\lambda} e^{-\bar\lambda\bar z_1} \frac{\partial\psi}{\partial\bar z_1}(z,\lambda) \stackrel{\text{def}}{=} b(\lambda) = \bar\psi(z^*,\lambda)^{-1}\frac{\partial\psi}{\partial\bar\lambda}(z^*,\lambda),$$

$\forall\ z \in X$ function $\lambda \mapsto \psi(z,\lambda)$, $\lambda \in \mathbb{C}$, is a unique solution of the integral equation

$$\psi(z,\lambda) = e^{\lambda z_1} - \frac{1}{2\pi i}\int\limits_{\xi\in\mathbb{C}} b(\xi)e^{(\lambda-\xi)z_1}\bar\psi(\xi,\lambda)\frac{d\xi\wedge d\bar\xi}{\xi-\lambda}.$$

This proposition is a consequence of Propositions 2.1-2.3.

Proposition 3.3 (Reconstruction formulas for σ)
Conductivity function $\sigma\big|_X$ can be reconstructed through Dirichlet-Neumann data by the R.Novikov's scheme:

$$\text{DN data } \to \psi\big|_{bX} \to \bar\partial - \text{scattering data} \to \psi\big|_X \to \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\big|_X.$$

The last step of this scheme one can realize by any of the following formulas:

$$\text{A) } \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}(z) = (dd^c\psi(z,\lambda))\psi^{-1}(z,\lambda), \quad z \in X,\ \lambda \in \mathbb{C},$$

$$\text{B) } \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}(z) = 2i \lim_{\lambda\to\infty} \lambda e^{-\lambda z_1} dz_1 \wedge \bar\partial\psi(z,\lambda),\ z \in X,$$

$$\text{C) } \frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}(z) = -2i \lim_{\lambda\to 0} \frac{\bar\partial\partial\mu(z,\lambda)}{\mu(z,\lambda)},\ z \in X.$$

Proof
The first three steps of the scheme above were done in Propositions 3.1, 3.2. The formula A) is an immediate consequense of equation (3.1). The formula B) follows from equation $\bar\partial(\partial + \lambda dz_1)\mu = \frac{i}{2}\frac{dd^c\sqrt{\sigma}}{\sqrt{\sigma}}\mu$ (Proposition 1.2a) and estimates $\mu \to 1$, $\lambda \to \infty$ (Proposition 1.2b), $\partial\bar\partial\mu(z,\lambda) \to 0$, $\lambda \to \infty$ (Proposition 1.2e). The formula C) follows from the same equation and estimates $\|\frac{\partial\mu}{\partial z_1}\| = O(\sqrt{|\lambda|})$, $\lambda \to 0$, (Proposition 1.2c).

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